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General brane cosmology with ${}^{(4)}R$ term in $(A)dS_5$ or Minkowski bulk

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Abstract: A general analysis of the induced brane dynamics is performed in an $(A)dS_5$ or Minkowski background, when the intrinsic curvature term is included in the brane action. Such a term is known to cause dramatic changes and is generically induced by quantum corrections coming from the bulk gravity and its coupling with matter living on the brane. The induced brane dynamics is shown to be the usual Einstein dynamics coupled to a well-defined modified energy momentum tensor. Conventional general relativity revives for an initial era whose duration depends on the value of the five-dimensional Planck mass. Violations of energy conditions may be possible, as well as matter inhomogeneities on the brane in the above exact bulks. A new anisotropic solution is given in the above context. This solution, for a fine-tuned AdS_5 bulk, exhibits an intermediate accelerating phase which is followed by an era corresponding to a 4D perfect fluid solution with no future horizons.

Keywords: Extra Large Dimensions, p-branes, Classical Theories of Gravity.

Contents

1. Introduction

Brane cosmologies consist cosmological realizations of string theories in which some underlying features are often minimized. Ordinary matter fields living on the brane are represented by open-string excitations, while gravitons by closed string loops, which not being trapped on the brane, enable the gravitons to penetrate into the higher dimensional bulk. Thus, gravity is fundamentally a higher dimensional effect and the bulk Einstein equations are generally used to deduce the dynamics on the brane. For a given bulk metric, the induced metric on a specific brane is certainly uniquely defined. Since there is basically no preferred position for a brane in a given bulk, the knowledge of the particular induced metric is irrelevant. However, the knowledge of the induced brane gravitational dynamics, irrespectively of the brane position, becomes a fact of particular importance, whenever this is possible. In an $(A)dS$ bulk (or even Minkowski), when the brane has codimension one and the effective low-energy theory in the bulk is higher dimensional gravity, such a reduction is possible [1]. A more fundamental description of the physics that produces the brane could include [2] higher order terms in a derivative expansion of the effective action, such as a term for the scalar curvature of the brane, and higher powers of curvature tensors on the brane. A brane action that contains powers of the brane curvature tensors has also been used in the context of the AdS/CFT correspondence (e.g. [3]) to regularize the action of a bulk AdS space which diverges when the radius of the AdS space becomes infinite. If the dynamics is governed not only by the ordinary five-dimensional Einstein-Hilbert action, but also by the four-dimensional Ricci scalar term induced on the brane, new phenomena appear. In [4, 5] it was observed that the localized matter fields on the brane (which couple to bulk gravitons) can generate via quantum loops a localized four-dimensional worldvolume kinetic term

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for gravitons (see also [6]–[9]). That is to say, four-dimensional gravity is induced from the bulk gravity to the brane worldvolume by the matter fields confined to the brane. It was also shown that an observer on the brane will see correct Newtonian gravity at distances shorter than a certain crossover scale, despite the fact that gravity propagates in extra space which was assumed there to be flat with infinite extent. At larger distances, the force becomes higher-dimensional. The first realization of the induced gravity scenario in string theory was presented in [10]. Furthermore, new closed string couplings on Dp-branes for the bosonic string were found in [11]. These couplings are quadratic in derivatives and therefore take the form of induced kinetic terms on the brane. For the graviton in particular these are the induced Einstein-Hilbert term as well as terms quadratic in the second fundamental tensor. The inclusion of an intrinsic curvature term is naturally expected to affect the cosmological expansion of the Universe. In [12]–[16] the Friedmann-like equations for particular cosmological examples were discussed in this context. In [15] two isotropic cosmological solutions were found (in Minkowski bulk) close to the usual FRW cosmology for small enough Hubble radius, while far beyond the cross-over scale, the Universe enters a fully 5D regime or to a self-inflationary expansion arising without including any effective cosmological constant on the brane. With five-dimensional Planck mass M_5 being of the order of TeV [17] the above scenario seems phenomenologically not very viable at least from a cosmological and astronomical perspective [4]. However, in [18] a TeV string scale together with one extra compact dimension of astronomical size can circumvent all phenomenological difficulties. Further, in [19] a framework with an infinite-volume flat 5D space was proposed in which the quantum gravity scale can be as low as 10^{-3} eV without conflicting with any of the existing laboratory, astrophysical or cosmological bounds. An observer on the brane can see conventional gravity from a lower distance up to astronomically large distances; outside this range gravity becomes five-dimensional. If $M_5 \sim 10$MeV, then [20, 15] a self-accelerated 4D universe predicts modification of gravitational laws at scales comparable with the present cosmological horizon and can be in agreement with the recent Supernovae and acoustic peaks of Cosmic Microwave Background radiation. Earlier discussions on lowering the string scale relative to M_4 were given in [21, 22]. In [10] a relevant analysis with ${}^{(4)}R$ term introduced a new threshold scale, due to the width of a non-zero thickness brane; at distances larger than this scale physics is four-dimensional, while at shorter distances irregular deviations appear. Systems similar to the one studied in the present paper have been also studied in [23] where some discussion of the role played by the induced curvature term within the context of the AdS/CFT correspondence has been given. The situation with an Einstein curvature term added to the brane action, but without imposing the reflectional symmetry between the two sides of the brane, has been discussed in [24]

In the present work we are treating the general relativistic case with no restrictions on the metric form, when the above intrinsic curvature term is included. The background is assumed to be $(A)dS$, but a Minkowski bulk is also involved. The additional term is geometrical in nature and thus creates modified geometrical terms beyond the Einstein ones in the induced (brane) equations. This is the line of work followed in the relevant cosmological models [12]–[15] and makes the further investigation of the induced dynamics difficult. We show that these induced equations can assume a usual Einstein equation form with additional matter content. This enables us to treat the brane dynamics with the methods of conventional general relativity. More specifically, the energy momentum tensor in these equations is split into the common brane energy momentum tensor plus additional terms which are all multiplied by one of the characteristic scales of the theory, M_5^3/M_4^2. Conventional general relativity revives for an initial era which depends on the value of the above scale. An ordinary matter on the brane can produce terms in these effective equations which may cause violations of the energy conditions. Such violations arising from a different context also appeared in [25]. We have performed major part of the analysis in $n+1$ spacetime dimensions. In an $(A)dS$ or Minkowski bulk the conservation of energy on the brane does not necessarily imply the existence of only spatially homogeneous universes (as this happens when ${}^{(4)}R$ is not included [1]), i.e. inhomogeneous perfect fluid brane solutions might exist in the above exact bulks. In section 3, we incorporate the above formulation to the class of four-dimensional spatially homogeneous spacetimes and find, as an example, a new anisotropic four-dimensional solution. There exists an AdS_5 bulk such that this solution possesses an accelerating phase after the conventional four-dimensional evolution, followed by another 4D perfect fluid solution with no event horizons. Since this contains the parameter M_5, one could try to compare this with the well-known value of the quadrupole moment of the CMB radiation and thus set restrictions on the value of the string scale. Finally, in section 4, we conclude and speculate on possible generalizations.

2. General analysis with intrinsic curvature Ricci scalar

We start with a $(n+1)$-dimensional theory and a $(n-1)$-brane Σ embedded in $(n+1)$-dimensional spacetime M. Capital latin letters $A, B, \ldots = 1, \ldots, n+1$ will denote full spacetime, lower Greek $\mu, \nu, \ldots = 1, \ldots, n$ run over brane worldvolume, while lower Latin ones span some $(n-1)$-dimensional spacelike surfaces foliating the brane, i.e. $i, j, \ldots = 1, \ldots, n-1$. For convenience, we can quite generally, choose a coordinate y such that the hypersurface $y = 0$ coincides with the brane. Our primary interest lies in $n = 4$ where the brane is supposed to represent our universe. The total action for the system is taken to be:

$$
\begin{aligned}
S = & \frac{1}{2\kappa_{n+1}^2} \int_M \sqrt{\varepsilon_{n+1}\, {}^{(n+1)}g}\; \left({}^{(n+1)}R - 2\Lambda_{n+1}\right) d^{n+1}x + \\
& + \frac{1}{2\kappa_n^2} \int_\Sigma \sqrt{\varepsilon_n\, {}^{(n)}g}\; \left({}^{(n)}R - 2\Lambda_n\right) d^n x +
\end{aligned}
$$

$$+\int_M \sqrt{\varepsilon_{n+1}\,{}^{(n+1)}g}\; L^{mat}_{n+1}\, d^{n+1}x + \int_\Sigma \sqrt{\varepsilon_n\,{}^{(n)}g}\; L^{mat}_n\, d^n x\,. \tag{2.1}$$

For clarity, we have separated the cosmological constants Λ_{n+1}, Λ_n from the rest matter contents L^{mat}_{n+1}, L^{mat}_n of the bulk and the brane respectively. Λ_n/κ_n^2 can be interpreted as the brane tension of the standard Dirac-Nambu-Goto action, or as the sum of a brane worldvolume cosmological constant and a brane tension. For possible treatment of different than usual signatures of the metrics we have allowed the signs ε_{n+1}, ε_n. We basically concern on an $(A)dS$ bulk, which means that the Weyl tensor vanishes and also ${}^{(n+1)}T_{AB}=0$. If the bulk is Minkowski, additionally $\Lambda_{n+1}=0$.

From the dimensionful constants κ^2_{n+1}, κ^2_n the Planck masses M_{n+1}, M_n are defined as:

$$\kappa^2_{n+1} = 8\pi G_{(n+1)} = M_{n+1}^{-(n-1)}\,, \qquad \kappa_n^2 = 8\pi G_{(n)} = M_n^{-(n-2)}\,, \tag{2.2}$$

with M_{n+1}, M_n having dimensions of (length)$^{-1}$. Then, a distance scale r_c is defined as :

$$r_c \equiv \frac{\kappa^2_{n+1}}{\kappa^2_n} = \frac{M_n^{n-2}}{M_{n+1}^{n-1}}\,. \tag{2.3}$$

Varying (2.1) with respect to the bulk metric g_{AB}, we obtain the equations

$${}^{(n+1)}G_{AB} = -\Lambda_{n+1} g_{AB} + \kappa^2_{n+1}\left({}^{(n+1)}T_{AB} + {}^{(loc)}T_{AB}\,\delta(y)\right), \tag{2.4}$$

where

$${}^{(loc)}T_{AB} \equiv -\frac{1}{\kappa_n^2}\sqrt{\frac{\varepsilon_n\,{}^{(n)}g}{\varepsilon_{n+1}\,{}^{(n+1)}g}}\;\left({}^{(n)}G_{AB} - \kappa_n^2\,{}^{(n)}T_{AB} + \Lambda_n g_{AB}\right) \tag{2.5}$$

is the localized energy-momentum tensor of the brane. ${}^{(n+1)}G_{AB}$, ${}^{(n)}G_{AB}$ denote the Einstein tensors constructed from the bulk and the brane metrics respectively. Clearly, ${}^{(n)}G_{AB}$ acts as an additional source term for the brane through ${}^{(loc)}T_{AB}$.

The way the y-coordinate has been defined allows us to write at least in the neighborhood of the brane the $(n+1)$-line element as:

$$ds^2_{(n+1)} = g_{AB}dx^A dx^B = \varepsilon_t N^2 dt^2 + g_{ij}dx^i dx^j + \varepsilon_y\, b^2 dy^2\,, \tag{2.6}$$

where N, g_{ij}, b are generally functions of t, x^i, y. The splitting of the brane metric into space and time parts allows the choice of zero shift in (2.6).

According to the metric (2.6), the Einstein equations (2.4) of the bulk space are split into the following set of equations of the canonical analysis [26]:

$$K^\nu_{\mu;\nu} - K_{;\mu} = \varepsilon_y \kappa^2_{n+1}\, b\;{}^{(n+1)}T^y_\mu \tag{2.7}$$

$$K^\mu_\nu K^\nu_\mu - K^2 + \varepsilon_y\,{}^{(n)}R = 2\varepsilon_y(\Lambda_{n+1} - \kappa^2_{n+1}\,{}^{(n+1)}T^y_y) \tag{2.8}$$

$$\begin{aligned} K^{\mu}_{\nu}{}' + bKK^\mu_\nu - \varepsilon_y b\;{}^{(n)}R^\mu_\nu + \varepsilon_y g^{\mu\lambda} b_{;\lambda\nu} = &-\varepsilon_y \kappa^2_{n+1} b\left({}^{(loc)}T^\mu_\nu - \frac{{}^{(loc)}T}{n-1}\,\delta^\mu_\nu\right)\delta(y) - \\ &-\varepsilon_y\kappa^2_{n+1} b\,{}^{(n+1)}T^\mu_\nu + \frac{\varepsilon_y b}{n-1}\times \\ &\times(\kappa^2_{n+1}\,{}^{(n+1)}T - 2\Lambda_{n+1})\delta^\mu_\nu\,, \end{aligned} \tag{2.9}$$

where $K_{AB} = h^C_A h^D_B \nabla_C n_D$ is the extrinsic curvature of the hypersurfaces y=constant, $h^A_B = \delta^A_B - \varepsilon_y n^A n_B$ is the projection operator, while n^A is the normal vector on these. In the above equations, K, ${}^{(n+1)}T$, ${}^{(loc)}T$ denote the traces of K^A_B, ${}^{(n+1)}T^A_B$, ${}^{(loc)}T^A_B$ (indices raised by g^{AB}), the semicolon stands for covariant differentiation with respect to the induced metric $g_{\mu\nu}$, while throughout, prime and dot mean partial derivatives with respect to y and t respectively.

Isolating the singular part of equations (2.9) we obtain the modified (due to ${}^{(n)}G^\mu_\nu$) Israel-Darmois-Lanczos-Sen conditions [27, 28, 29, 30]

$$[K^\mu_\nu] = -\varepsilon_y \kappa^2_{n+1} b_o \left({}^{(loc)}T^\mu_\nu - \frac{{}^{(loc)}T}{n-1}\delta^\mu_\nu \right), \tag{2.10}$$

where the bracket means discontinuity of the quantity across $y = 0$, and $b_o = b(y = 0)$. Hereafter, we consider a $\mathbb{Z}_2$ symmetry on reflection around the brane and thus (2.10) becomes

$${}^{(n)}G^\mu_\nu = \kappa^2_n \, {}^{(n)}T^\mu_\nu - \Lambda_n \delta^\mu_\nu - \varepsilon_y \alpha \left(\overline{K}{}^\mu_\nu - \overline{K}\delta^\mu_\nu \right), \tag{2.11}$$

where $\overline{K}{}^\mu_\nu = K^\mu_\nu(y = 0^+) = -K^\mu_\nu(y = 0^-)$ and $\alpha \equiv -2sgn(b_o)/r_c$.

These equations resemble Einstein equations on the brane but unfortunately, they contain the undetermined geometrical quantities $\overline{K}{}^\mu_\nu$. Fortunately however, we can do better based on a geometrical identity, namely Gauss equation, i.e.

$${}^{(n)}R^A_{BCD} = {}^{(n+1)}R^M_{NKL} h^A_M h^N_B h^K_C h^L_D + \varepsilon_y \left(K^A_C K_{BD} - K^A_D K_{BC} \right). \tag{2.12}$$

From the above relation, taking suitable contractions to construct the n, $(n+1)$-dimensional Einstein tensors, and making use of the bulk Einstein equations, we get

$$\begin{aligned} {}^{(n)}G_{AB} = {} & \varepsilon_y \left(K K_{AB} - K_{AC} K^C_B \right) + \frac{\varepsilon_y}{2}(K^C_D K^D_C - K^2) h_{AB} - \frac{n-2}{2}\Lambda_{n+1} h_{AB} + \\ & + \frac{n-2}{n-1}\kappa^2_{n+1} \left({}^{(n+1)}T_{CD} h^C_A h^D_B + \left(\varepsilon_y \, {}^{(n+1)}T_{CD} n^C n^D - \frac{1}{n} \, {}^{(n+1)}T^C_C \right) h_{AB} \right) - \\ & - \varepsilon_y C^C_{EDH} n_C n^D h^E_A h^H_B \, , \end{aligned} \tag{2.13}$$

where C^A_{BCD} is the Weyl tensor of ${}^{(n+1)}R^A_{BCD}$. The parallel to our brane, components of the preceding equations give

$$\begin{aligned} {}^{(n)}G^\mu_\nu = {} & \varepsilon_y \left(\overline{K}\,\overline{K}{}^\mu_\nu - \overline{K}{}^\mu_\lambda \overline{K}{}^\lambda_\nu \right) + \frac{\varepsilon_y}{2} \left(\overline{K}{}^\kappa_\lambda \overline{K}{}^\lambda_\kappa - \overline{K}{}^2 \right)\delta^\mu_\nu - \frac{n-2}{n}\Lambda_{n+1}\delta^\mu_\nu + \\ & + \frac{n-2}{n-1}\kappa^2_{n+1} \left({}^{(n+1)}\overline{T}{}^\mu_\nu + \left({}^{(n+1)}\overline{T}{}^y_y - \frac{{}^{(n+1)}\overline{T}}{n} \right) \delta^\mu_\nu \right) - g^{\kappa\mu}\,\overline{C}{}^y_{\kappa y\nu} \, , \end{aligned} \tag{2.14}$$

where we have put a bar over ${}^{(n+1)}T^A_B$, $C^y_{\kappa y\nu}$ to show explicitly that all the quantities in (2.14) are evaluated at $y = 0$. Equations (2.13), (2.14) do not contain any singular part since the distributional part of the system has been extracted as the boundary condition for the bulk imposed by equation (2.11).

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Equations (2.14) are independent from (2.11), so we can get additional information on $\overline{K}{}^{\mu}_{\nu}$ by equating their right hand sides:

$$\overline{K}{}^{\mu}_{\lambda}\overline{K}{}^{\lambda}_{\nu} - \overline{K}\,\overline{K}{}^{\mu}_{\nu} + \frac{1}{2}\,(\overline{K}{}^{2} - \overline{K}{}^{\kappa}_{\lambda}\overline{K}{}^{\lambda}_{\kappa})\,\delta^{\mu}_{\nu} - \alpha(\overline{K}{}^{\mu}_{\nu} - \overline{K}\delta^{\mu}_{\nu}) = \mathcal{T}^{\mu}_{\nu}\,, \tag{2.15}$$

where

$$\mathcal{T}^{\mu}_{\nu} = \varepsilon_y\left(\Lambda_n - \frac{n-2}{n}\,\Lambda_{n+1}\right)\delta^{\mu}_{\nu} - \varepsilon_y\,\kappa_n^2\,{}^{(n)}T^{\mu}_{\nu} +$$
$$+\varepsilon_y\,\frac{n-2}{n-1}\,\kappa_{n+1}^2\left({}^{(n+1)}\overline{T}{}^{\mu}_{\nu} + \left({}^{(n+1)}\overline{T}{}^{y}_{y} - \frac{{}^{(n+1)}\overline{T}}{n}\right)\delta^{\mu}_{\nu}\right) - \varepsilon_y\,g^{\kappa\mu}\,\overline{C}{}^{y}_{\kappa y\nu}\,. \tag{2.16}$$

We could drop the Weyl tensor from the above expression and set ${}^{(n+1)}\overline{T}{}^{A}_{B} = 0$ since we basically concern on an $(A)dS_{n+1}$ or Minkowski bulk. We just keep them temporarily for the case that some other bulk space will acquire in the future a favored position; the relevant induced terms could be interpreted as additional source terms.

Equation (2.15) is an algebraic equation for $\overline{K}{}^{\mu}_{\nu}$ in terms of $\mathcal{T}^{\mu}_{\nu}$, which we will try to solve. Taking the trace of (2.15) and plugging back in the same equation, we obtain equivalently

$$\overline{K}{}^{\mu}_{\lambda}\overline{K}{}^{\lambda}_{\nu} - (\overline{K}+\alpha)\overline{K}{}^{\mu}_{\nu} - \frac{\alpha}{n-2}\overline{K}\,\delta^{\mu}_{\nu} = \mathcal{T}^{\mu}_{\nu} - \frac{\mathcal{T}^{\lambda}_{\lambda}}{n-2}\,\delta^{\mu}_{\nu}\,. \tag{2.17}$$

Now, setting

$$L^{\mu}_{\nu} \equiv \overline{K}{}^{\mu}_{\nu} - \frac{\overline{K}+\alpha}{2}\,\delta^{\mu}_{\nu}\,, \tag{2.18}$$

equation (2.17) gets the form

$$L^{\mu}_{\lambda}L^{\lambda}_{\nu} - \frac{L^2}{(n-2)^2}\,\delta^{\mu}_{\nu} = \mathcal{T}^{\mu}_{\nu} - \frac{(n-1)\alpha^2 + (n-2)\mathcal{T}^{\lambda}_{\lambda}}{(n-2)^2}\,\delta^{\mu}_{\nu}\,, \tag{2.19}$$

where $L \equiv L^{\mu}_{\mu}$. Supposed that (2.19) has been solved, substituting $\overline{K}{}^{\mu}_{\nu}$ back in (2.11) in terms of the matter contents $\mathcal{T}{}^{\mu}_{\nu}$, we will have strictly a system of Einstein equations for the brane metric with additional matter terms besides the conventional ones, i.e.

$${}^{(n)}G^{\mu}_{\nu} = \kappa_n^2\,{}^{(n)}T^{\mu}_{\nu} - \left(\Lambda_n + \varepsilon_y\,\frac{n-1}{n-2}\alpha^2\right)\delta^{\mu}_{\nu} - \varepsilon_y\alpha\left(L^{\mu}_{\nu} + \frac{L}{n-2}\,\delta^{\mu}_{\nu}\right). \tag{2.20}$$

An effective cosmological constant arises which exists even for $\Lambda_n = 0$. The various terms appeared in the right hand side of equation (2.20) consist the effective energy-momentum tensor, which presumably can violate some energy conditions, even if the conventional matter terms do not. When M_4 in the action (2.1) becomes much larger than M_5, then $\alpha \to 0$ and the above equation reduces to the expected four-dimensional General Relativity (at least whenever the quantity multiplying α does not diverge).

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Due to the block-diagonal form of the n-part of metric (2.6), we have that $\overline{K}{}^{\,0}_{i} = \overline{K}{}^{\,i}_{0} = 0$ and thus $L^0_i = L^i_0 = 0$. Then, from (2.19) it arises that $\mathcal{T}_i^{\,0} = \mathcal{T}_0^{\,i} = 0$. Furthermore, the system (2.19) can be decomposed into the following set of equations:

$$L^i_l L^l_j = ((L^0_0)^2 - \mathcal{T}_0^{\,0})\,\delta^i_j + \mathcal{T}^i_j\,, \tag{2.21}$$

$$L^i_i = -L^0_0 \pm \sqrt{(n-2)^2(L^0_0)^2 - (n-2)(n-3)\mathcal{T}_0^{\,0} + (n-2)\mathcal{T}_i^{\,i} + (n-1)\alpha^2}\;. \tag{2.22}$$

The unknown matrices L^i_j in (2.21) are $(n-1)$-dimensional and if solved in terms of L^0_0 and $\mathcal{T}_\nu^{\,\mu}$, then (2.22) will set an algebraic equation for L^0_0. In order for a solution to exist, it has to be $((L^0_0)^2 - \mathcal{T}_0^{\,0})\delta^i_j + \mathcal{T}^i_j$ and the quantity under the square root of (2.22) non-negative.

To assure the existence of solutions for equation (2.21), we assume the situation where $\mathcal{T}^i_j$ has $n-1$ (real) eigenvalues $\tau_1, \ldots, \tau_{n-1}$, not necessarily distinct and its minimal polynomial has simple roots only. (Of course, the case with $\mathcal{T}^i_j$ having $n-1$ distinct eigenvalues is also included). Then, there exists some invertible (x^μ-dependent) matrix P^i_j which diagonalizes L^i_j, i.e.

$$L^i_j = (P^{-1})^i_k\, \widetilde{L}^k_l\, P^l_j\,, \tag{2.23}$$

where the components in this new frame are

$$\widetilde{L}^i_j = \mathrm{diag}(\,\pm\sqrt{(L^0_0)^2 + \tau_1 - \mathcal{T}_0^{\,0}}\;, ..., \pm\sqrt{(L^0_0)^2 + \tau_{n-1} - \mathcal{T}_0^{\,0}}\,)\,. \tag{2.24}$$

The various $\pm$ appeared are independent each other. Then, equation (2.22) gives

$$\pm\sum_{i=1}^{n-1} \sqrt{(L^0_0)^2 + \tau_i - \mathcal{T}_0^{\,0}} \;\pm\; L^0_0 = \tag{2.25}$$

$$= \sqrt{(n-2)^2(L^0_0)^2 - (n-2)(n-3)\mathcal{T}_0^{\,0} + (n-2)\sum_{i=1}^{n-1}\tau_i + (n-1)\alpha^2}$$

with all the $\pm$'s being independent. The above equation is not easily solved for L^0_0 in terms of the matter, but if this is done, then (2.20) become well-defined Einstein equations for the brane with modified energy-momentum tensor.

Of particular importance is the subcase with $\mathcal{T}^i_j$ isotropic, i.e. $\mathcal{T}^i_j = \tau\,\delta^i_j$ (as it is seen from (2.16) this happens, for example, when the brane and/or the bulk contain untilted perfect fluids, for zero Weyl tensor in the bulk). Then, the solution of (2.21) given by (2.23) and (2.24) is

$$L^i_j = S\,(P^{-1})^i_k\, E^k_l\, P^l_j\,, \tag{2.26}$$

where

$$S = \sqrt{(L^0_0)^2 + \tau - \mathcal{T}_0^{\,0}}\,, \tag{2.27}$$

$$E^i_j = \mathrm{diag}(+1, \ldots, +1, -1, \ldots, -1)\,, \tag{2.28}$$

with $n_+ \geq 0$ signs $+1$, $n_- \geq 0$ signs -1, and $n_+ + n_- = n-1$. We have to discern two cases.

Let $\underline{n_+, n_- \neq 0, 1}$. Then, equation (2.22) (or equivalently (2.25)) is a quadratic for $(L_0^0)^2$ and supplies the following explicit solution for L_0^0:

$$L_0^0 = \oplus \frac{1}{\sqrt{8n_+n_-(n_+-1)(n_--1)}} [-2(n_+-1)(n_--1)(n-1-4n_+n_-)\mathcal{T}_0^0 + \\ +2n_+n_-(3n-5-4n_+n_-)\tau + (n-1)(n-1-2n_+n_-)\alpha^2 \pm |n_+-n_-|\, A]^{\frac{1}{2}} \quad (2.29)$$

where

$$A = \ [\, -4(n_+-1)(n_--1)\mathcal{T}_0^0 \ (\,(n-2)\mathcal{T}_0^0 + (n-1)\alpha^2\,) + \\ + 4n_+n_-\tau \ (\,(n-2)\tau + (n-1)\alpha^2\,) \ + \ (n-1)^2\alpha^4 \ ]^{1/2}\,. \quad (2.30)$$

The quantity S in (2.26) becomes

$$S = \frac{1}{\sqrt{8n_+n_-(n_+-1)(n_--1)}} [-2(n_+-1)(n_--1)(n-1)\mathcal{T}_0^0 - 2n_+n_-(n-3)\tau + \\ + (n-1)(n-1-2n_+n_-)\alpha^2 \pm |n_+-n_-|\, A\,]^{1/2}\,. \quad (2.31)$$

It is assumed that the matter contents are such that the square roots appeared in (2.29), (2.30), (2.31) and the right hand side of (2.25) are all well-defined. The sign $\oplus$ in (2.29) means that an overall + or − sign can be taken independently of the other $\pm$ of (2.29), (2.31) which however go together.

Let $\underline{\text{some of } n_+, n_- \text{ take the value 0 or 1}}$. (Obviously, the case of our primary interest $n=4$ is included here). Then, (2.22) (or (2.25)) has vanishing coefficient of $(L_0^0)^4$ and gives :

$$L_0^0 = \pm\frac{1}{2}\,[(3n-5-4n_+n_-)\mathcal{T}_0^0 - (n-1-4n_+n_-)\tau + (n-1)\alpha^2]\,/B\,, \quad (2.32)$$

where

$$B = [-2(n_+-1)(n_--1)(n-1)\mathcal{T}_0^0 + 2\,n_+n_-(1-n_+n_-)\tau + \\ + (n-1)(n-1-2n_+n_-)\alpha^2\,]^{1/2}\,, \quad (2.33)$$

and from (2.27):

$$S = \frac{1}{2}\,|(n-3)\mathcal{T}_0^0 + (n-1)(\tau+\alpha^2)|\,/B\,. \quad (2.34)$$

In this case, we are remained with the square root of (2.33) and that of the right hand side of (2.25) to be well-defined.

We observe that since n_+, n_- insert equations (2.29)–(2.34) in a symmetric way, the interchange of the number of +1's with that of −1's does not affect the equations.

Gathering together the above results for the isotropic case, we rewrite equations (2.20) for the previously found L^0_0, S (equations (2.29), (2.31), (2.32), (2.34)) as:

$$ {}^{(n)}G^0_0 = \kappa_n^2\, {}^{(n)}T^0_0 - \left(\Lambda_n + \varepsilon_y \frac{n-1}{n-2}\,\alpha^2\right) - \varepsilon_y \alpha \frac{n-1}{n-2}\left(L^0_0 + \frac{n_+ - n_-}{n-1}\,S\right), \tag{2.35} $$

$$ {}^{(n)}G^i_j = \kappa_n^2\, {}^{(n)}T^i_j - \left(\Lambda_n + \varepsilon_y \frac{n-1}{n-2}\,\alpha^2\right)\delta^i_j - \varepsilon_y \alpha S E^i_j - \frac{\varepsilon_y \alpha}{n-2}\left(L^0_0 + (n_+ - n_-)S\right)\delta^i_j\,, \tag{2.36} $$

where now the indices i, j refer to the new frame. Thus, the inclusion of the term ${}^{(n)}R$, instead of creating additional difficulties, has brought a convenient decomposition of the matter terms. First, standard energy-momentum tensor enters without having made (as in [31, 1]) any choice for the brane tension Λ_4 in terms of M_4, M_5. Note that if ${}^{(4)}R$ is not included in the action, then for $\Lambda_4 = 0$, ordinary energy terms cannot arise. Furthermore, in that case, Λ_4 has to be positive in order for κ_4^2 to be positive. Second, the additional matter terms (which rather appear here as square roots instead of squares of the four-dimensional energy-momentum tensor) are *all* multiplied by the energy scale α. We will come back on this later on, but this already sounds promising in reviving standard general relativistic cosmologies, by simply constraining the value of Planck mass M_5. Without including the intrinsic curvature term, conventional four-dimensional Einstein gravity arises in the low energy world, when additionally to the above referred condition for Λ_4, the characteristic energy scale of the matter, $\kappa_4^2\, {}^{(4)}T^\mu_\nu$, is much lower than Λ_4, and also Λ_5, Λ_4 are fine-tuned [31, 1]. However, in [1] it was stated that if simply $\kappa_5 \to 0$, keeping $\kappa_5^4 \Lambda_4$ finite, common four-dimensional gravity arises. This is not exactly correct, since under the above conditions, the brane constant Λ_4 does not enter the induced Einstein equations as it is, but multiplied by $4\pi G_{(4)}$.

If we concentrate on the case $n = 4$, there are only two different situations: (i) $n_+ = 3$, $n_- = 0$, and (ii) $n_+ = 2$, $n_- = 1$.
(i) From (2.32), (2.34) we have

$$ L^0_0 = \pm \frac{1}{2\sqrt{3}}\, \frac{7\mathcal{T}^0_0 - 3\tau + 3\alpha^2}{\sqrt{4\mathcal{T}^0_0 + 3\alpha^2}}\,, \qquad S = \frac{1}{2\sqrt{3}}\, \frac{|\mathcal{T}^0_0 + 3\tau + 3\alpha^2|}{\sqrt{4\mathcal{T}^0_0 + 3\alpha^2}}\,. \tag{2.37} $$

(ii) Similarly, we obtain

$$ L^0_0 = \pm \frac{1}{2}\, \frac{-\mathcal{T}^0_0 + 5\tau + 3\alpha^2}{\sqrt{-4\tau - 3\alpha^2}}\,, \qquad S = \frac{1}{2}\, \frac{|\mathcal{T}^0_0 + 3\tau + 3\alpha^2|}{\sqrt{-4\tau - 3\alpha^2}}\,. \tag{2.38} $$

The only conditions which have to be satisfied here are those concerning the square roots of the denominators in (2.37), (2.38).

Taking the jump across $y=0$ of the linear equations (2.7) and using (2.10), we obtain the usual conservation equations for the brane

$$ {}^{(n)}T^{\mu}_{\nu\,;\,\mu} = 0\,, \tag{2.39} $$

just like the case where the ${}^{(4)}R$ term was not present (assuming that there is no energy flow from the brane towards the bulk and vice-versa, i.e. ${}^{(n+1)}T^{y}_{\mu} = 0$). From equations (2.20) written in the new frame, (2.39), and the contracted Bianchi identities ${}^{(n)}G^{\mu}_{\nu\,;\,\mu}=0$, it arises that

$$ L^{\mu}_{\nu\,;\,\mu} + \frac{L_{\,;\,\nu}}{n-2} = 0\,. \tag{2.40} $$

Whenever L^{μ}_{ν} has been found in terms of the matter, the above equations (2.40) will impose restrictions on the form of the matter admitted. Suppose that the bulk space is exactly $(A)dS_5$ or Minkowski. Then, equations (2.40) are written equivalently as:

$$ 3\dot{L}^{0}_{0} + (n_{+}-n_{-})\dot{S} + \frac{\dot{\gamma}}{\gamma}L^{0}_{0} - Sg^{jl}\dot{g}_{li}E^{i}_{j} = 0\,, \tag{2.41} $$

$$ L^{0}_{0\,,\,i} + (n_{+}-n_{-})S_{,i} + 2S_{,j}E^{j}_{i} + 2\frac{N_{,j}}{N}\,(SE^{j}_{i} - L^{0}_{0}\delta^{j}_{i}) + 2S\,(\,{}^{(3)}\Gamma^{j}_{lj}E^{l}_{i} - {}^{(3)}\Gamma^{l}_{ij}E^{j}_{l}\,) = 0\,, \tag{2.42} $$

where $L^{0}_{0}\,, S$ are given by (2.37) or (2.38).

If the matter content of the brane is an untilted perfect fluid, i.e. ${}^{(n)}T_{\mu\nu} = (\rho+p)u_{\mu}u_{\nu}+pg_{\mu\nu}$, where $u^{0}=\frac{1}{N}$ (all other components of u^{A} are zero) and $p=p(\rho)$ is the equation of state, then (2.39) is known for metric (2.6) to be written equivalently as:

$$ \dot{\rho} + (\rho+p)\frac{\dot{\gamma}}{2\gamma} = 0\,, \tag{2.43} $$

$$ p_{,i} + (\rho+p)\frac{N_{,i}}{N} = 0\,, \tag{2.44} $$

where $\gamma \equiv det(g_{ij})$. Substituting $\frac{\dot{\gamma}}{\gamma}$ and $\frac{N_{,i}}{N}$ from (2.43), (2.44) into (2.41), (2.42) respectively, we obtain one equation for $\dot{\rho}$ and one for $\rho_{,i}$. Then, for the case (i) with $E^{i}_{j}=\delta^{i}_{j}$, it can be easily seen that $\rho_{,i}=0$ for any equation of state. This means that in this case, if one wishes to describe inhomogeneous perfect fluids, has to depart from exact $(A)dS_5$ or Minkowski bulks. Inhomogeneity in the observable universe is one way to describe temperature anisotropies in the CMB spectrum (e.g. through the Sachs-Wolfe effect [32]). In the second case (ii) with $E^{i}_{j}=\text{diag}(+1,+1,-1)$, the above equations for $\dot{\rho}$, $\rho_{,i}$ do not necessarily imply vanishing $\rho_{,i}$. If these equations are solved for $\rho(t,x^{i})$ under some conditions for the geometry and for some equation of state, then equations (2.43), (2.44) and the other dynamical Einstein equations will presumably supply the whole cosmological solution. It is probable that the equation of state should not be given a priori, but the compatibility of the system would supply some rather complicated equation of state (as e.g. the solution given in [33]). Thus, even we have not proven the existence of brane inhomogeneous solutions in an $(A)dS_5$ or Minkowski bulk, we have shown that the conservation equations of the common matter does not exclude such a possibility. This is in contrast to the usual brane cosmologies (without including ${}^{(4)}R$), where the conservation of energy on the brane in a $(A)dS_5$ or Minkowski bulk enforces the existence of only spatially homogeneous universes [1].

One can, at this point, look at the right hand side of equations (3.8), (3.9) of next section, where V and G are given by (3.10)–(3.13). These are the matter terms appeared for a perfect fluid of case (i) (equations (2.37)) in the full general relativity case (not only in the spatially homogeneous cases of next section). In the case where ${}^{(4)}R$ is not included, in the past history of a universe, i.e. when $\kappa_4^2\rho \gg \alpha^2, \Lambda_4, \Lambda_5$, the dominant matter terms are $\kappa_4^2\rho$, $\kappa_4^2 p$. Including ${}^{(4)}R$, the additional term $\alpha\kappa_4\sqrt{\rho}$ emerges (roughly speaking this is correct when the equation of state has $p/\rho = w$ =constant). But, in the above era of matter domination, this last term is too small relative to the previous ones and can be ignored. Thus, we are remained with common general relativity without any modified matter terms. To estimate the interval of validity of this approximation we integrate equation (2.43) for $p = w\rho$ $(-1 < w \leq 1)$, i.e. $\rho = \rho_1(x^i)\,\gamma^{-\frac{w+1}{2}}$ (ρ_1 integration function), and then it arises that one must have $\gamma \ll (\frac{\kappa_4^2\rho_1}{\alpha^2})^{\frac{2}{w+1}}$. This value supplies an upper bound estimate of the volume scale factor of the universe for the common dynamics to be valid. Obviously, if M_5 is small enough, we can revive standard dynamics up to any desired cosmological scale. The same arguments are also seen to be true for case (ii).

3. An anisotropic example

Since we have found the Einstein equations (2.20) governing the modified dynamics of the brane, we can attempt to find new classes of brane solutions. Due to the metric (2.6), the system (2.20) is equivalent to the following set of equations (in the rotated basis supposed that it is a coordinate one):

$$k^i_{j\,|\,i} - (k^i_i)_{\,|\,j} = \varepsilon_t\,\kappa_n^2\,N\ {}^{(n)}T^0_j\,, \qquad k^i_j \equiv \frac{1}{2N}\,g^{il}\dot{g}_{lj}\,, \tag{3.1}$$

$$\begin{aligned}\dot{g}^{ij}\dot{g}_{ij} + \left(\frac{\dot{\gamma}}{\gamma}\right)^2 - 4\varepsilon_t N^2\ {}^{(n-1)}R = 8\varepsilon_t N^2\kappa_n^2\ {}^{(n)}T^0_0 - 8\varepsilon_t N^2\left(\Lambda_n + \varepsilon_y\frac{n-1}{n-2}\alpha^2\right) - \\ - \frac{8\varepsilon_t\varepsilon_y\alpha}{n-2}N^2\ \left((n-1)L^0_0 + L^i_i\right),\end{aligned} \tag{3.2}$$

$$\ddot{g}_{ij} + \left(\frac{\dot{\gamma}}{2\gamma} - \frac{\dot{N}}{N}\right)\dot{g}_{ij} - g^{kl}\dot{g}_{ki}\dot{g}_{lj} - 2\varepsilon_t N^2\ {}^{(n-1)}R_{ij} =$$

$$= -2\varepsilon_t N N_{|ij} - 2\varepsilon_t N^2 \kappa_n^2 \left({}^{(n)}T_{ij} - \frac{{}^{(n)}T}{n-2} g_{ij} \right) + 2\varepsilon_t \varepsilon_y \alpha N^2 L_i^l \, g_{lj} -$$

$$- \frac{2\varepsilon_t \varepsilon_y n\alpha}{(n-2)^2} N^2 \left(L_0^0 + L_l^l \right) g_{ij} - \frac{4\varepsilon_t}{n-2} N^2 \left(\Lambda_n + \varepsilon_y \frac{n-1}{n-2} \alpha^2 \right) g_{ij} \,, \quad (3.3)$$

where $|$ stands for the covariant differentiation with respect to g_{ij}.

If the $(n-1)$-dimensional spacelike surfaces foliating the brane are homogeneous spaces, i.e. there exists a $(n-1)$-dimensional isometry group of motions acting on each such surface, then we have a spatially homogeneous brane. More precisely, the group is assumed to be simply connected and the spacelike surfaces can be identified with the group by singling out a point on these as the identity of the group. In this case, there are $n-1$ basis one-forms $\sigma^{\hat{\alpha}}(x^i)$, $\hat{\alpha} = 1, \ldots, n-1$, such that $d\sigma^{\hat{\alpha}} = -C^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}} \sigma^{\hat{\beta}} \wedge \sigma^{\hat{\gamma}}$ with $C^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}}$ being the structure constants of the corresponding isometry group [34]. Furthermore, a general homogeneous tensor field (i.e. invariant under the motion of the isometry group) $\Omega^i_j(t,x)$ (e.g. g_{ij}, k^i_j) can be decomposed as $\Omega^{\hat{\alpha}}_{\hat{\beta}}(t) \, \sigma^i_{\hat{\alpha}}(x) \, \sigma^{\hat{\beta}}_j(x)$, where $\sigma^i_{\hat{\alpha}}$ is the inverse matrix of $\sigma^{\hat{\alpha}}_i$. In what follows we focus on the situation with $\mathcal{T}^i_j$ being isotropic and also for the case (i) of $n=4$ holding. Then, the previous equations (3.1), (3.2), (3.3) (for the most interesting case $\varepsilon_y = -\varepsilon_t = 1$) get the form:

$$k^{\hat{\mu}}_{\hat{\alpha}} C^{\hat{\nu}}_{\hat{\mu}\hat{\nu}} - k^{\hat{\mu}}_{\hat{\nu}} C^{\hat{\nu}}_{\hat{\alpha}\hat{\mu}} = -\frac{1}{2} \kappa_4^2 N \; {}^{(4)}T^0_{\hat{\alpha}} \,, \quad (3.4)$$

$$\dot{\gamma}^{\hat{\alpha}\hat{\beta}} \dot{\gamma}_{\hat{\alpha}\hat{\beta}} + \left(\frac{\dot{\tilde{\gamma}}}{\tilde{\gamma}} \right)^2 + 4N^2 \; {}^{(3)}R = -8N^2 \kappa_4^2 \; {}^{(4)}T^0_0 + 8N^2 \left(\Lambda_4 + \frac{3}{2} \alpha^2 \right) + $$
$$+ 12\alpha N^2 \left(L_0^0 + S \right) , \quad (3.5)$$

$$\ddot{\gamma}_{\hat{\alpha}\hat{\beta}} + \left(\frac{\dot{\tilde{\gamma}}}{2\tilde{\gamma}} - \frac{\dot{N}}{N} \right) \dot{\gamma}_{\hat{\alpha}\hat{\beta}} - \gamma^{\hat{\mu}\hat{\nu}} \dot{\gamma}_{\hat{\mu}\hat{\alpha}} \dot{\gamma}_{\hat{\nu}\hat{\beta}} + 2N^2 \; {}^{(3)}R_{\hat{\alpha}\hat{\beta}} =$$

$$= 2N^2 \kappa_4^2 \left({}^{(4)}T_{\hat{\alpha}\hat{\beta}} - \frac{{}^{(4)}T}{2} \gamma_{\hat{\alpha}\hat{\beta}} \right) + 2N^2 \left(\Lambda_4 + \frac{3}{2} \alpha^2 \right) \gamma_{\hat{\alpha}\hat{\beta}} + 2\alpha N^2 \left(L_0^0 + 2S \right) \gamma_{\hat{\alpha}\hat{\beta}} \,, \quad (3.6)$$

where $g_{ij}(t,x) = \gamma_{\hat{\alpha}\hat{\beta}}(t) \, \sigma^{\hat{\alpha}}_i(x) \, \sigma^{\hat{\beta}}_j(x)$, $\tilde{\gamma} = det(\gamma_{\hat{\alpha}\hat{\beta}})$ and L_0^0, S are given by (2.37).

For the spatially homogeneous branes we are discussing in this section, the unique conservation equation (2.43), for a perfect fluid matter content, is integrated to

$$\rho(t) = \rho_1 \, \tilde{\gamma}^{\,-\frac{w+1}{2}} \,, \quad (3.7)$$

where ρ_1 is a constant of integration. When, additionally, the bulk space is assumed $(A)dS_5$ (or Minkowski for $\Lambda_5 = 0$), the previous equations (3.5), (3.6) are equivalent to the following system:

$$\dot{\gamma}^{\hat{\alpha}\hat{\beta}} \dot{\gamma}_{\hat{\alpha}\hat{\beta}} + \left(\frac{\dot{\tilde{\gamma}}}{\tilde{\gamma}} \right)^2 + 4N^2 \; {}^{(3)}R = 4N^2 \left(\, 2\Lambda_4 + 3\alpha^2 + 2 \, \kappa_4^2 \, \rho + V(\tilde{\gamma}) \, \right) , \quad (3.8)$$

$$\ddot{\gamma}_{\hat{\alpha}\hat{\beta}} + \left(\frac{\dot{\tilde{\gamma}}}{2\tilde{\gamma}} - \frac{\dot{N}}{N}\right)\dot{\gamma}_{\hat{\alpha}\hat{\beta}} - \gamma^{\hat{\mu}\hat{\nu}}\dot{\gamma}_{\hat{\mu}\hat{\alpha}}\dot{\gamma}_{\hat{\nu}\hat{\beta}} + 2N^2\,{}^{(3)}R_{\hat{\alpha}\hat{\beta}} =$$

$$= N^2\left(\,2\Lambda_4 + 3\alpha^2 + \kappa_4^2\,(\rho - p) + G(\tilde{\gamma})\,\right)\gamma_{\hat{\alpha}\hat{\beta}}\,, \tag{3.9}$$

where

$$V(\tilde{\gamma}) = \pm\sqrt{3}\,\alpha\,\sqrt{4\Lambda_4 - 2\Lambda_5 + 3\alpha^2 + 4\kappa_4^2\,\rho}\,, \tag{3.10}$$

$$G(\tilde{\gamma}) = \pm\sqrt{3}\,\alpha\,\frac{4\Lambda_4 - 2\Lambda_5 + 3\alpha^2 + \kappa_4^2(3\rho - p)}{\sqrt{4\Lambda_4 - 2\Lambda_5 + 3\alpha^2 + 4\kappa_4^2\,\rho}}\,, \tag{3.11}$$

the above $+$ sign (resp. $-$) holding for the $+$ sign of equations (2.37) and $\mathcal{T}_0^0 + 3\tau + 3\alpha^2 > 0$ (resp. $-$ of (2.37) and $\mathcal{T}_0^0 + 3\tau + 3\alpha^2 < 0$), or

$$V(\tilde{\gamma}) = \pm\frac{3\sqrt{3}\,\alpha\,\kappa_4^2\,(\rho + p)}{\sqrt{4\Lambda_4 - 2\Lambda_5 + 3\alpha^2 + 4\kappa_4^2\,\rho}}\,, \tag{3.12}$$

$$G(\tilde{\gamma}) = \pm\,\alpha\,\frac{-4\Lambda_4 + 2\Lambda_5 - 3\alpha^2 + \kappa_4^2(5\rho + 9p)}{\sqrt{3}\,\sqrt{4\Lambda_4 - 2\Lambda_5 + 3\alpha^2 + 4\kappa_4^2\,\rho}}\,, \tag{3.13}$$

where now, the $+$ sign (resp. $-$) of (3.12), (3.13) holds for the $+$ sign of equations (2.37) and $\mathcal{T}_0^0 + 3\tau + 3\alpha^2 < 0$ (resp. $-$ of (2.37) and $\mathcal{T}_0^0 + 3\tau + 3\alpha^2 > 0$). Finally, equation (3.4) is written as

$$\gamma^{\hat{\beta}\hat{\delta}}\left(C^{\hat{\rho}}_{\hat{\alpha}\hat{\beta}}\dot{\gamma}_{\hat{\rho}\hat{\delta}} - C^{\hat{\rho}}_{\hat{\beta}\hat{\rho}}\dot{\gamma}_{\hat{\alpha}\hat{\delta}}\right) = 0\,. \tag{3.14}$$

The previous systems of equations can be solved as in conventional four-dimensional general relativity, by choosing some specific isometry group and adopting an ansatz for the three-metric $\gamma_{\hat{\alpha}\hat{\beta}}$. We will proceed giving an example, namely the anisotropic generalization of the open FRW universe, known as Bianchi type V geometry. This is characterized by the structure constants $C^{\hat{1}}_{\hat{1}\hat{3}} = C^{\hat{2}}_{\hat{2}\hat{3}} = 1/2$, other combinations vanish. Choose the temporal gauge $N = \sqrt{\tilde{\gamma}}$. Then, equations (3.9) become

$$\ddot{\gamma}_{\hat{\alpha}\hat{\beta}} - \gamma^{\hat{\mu}\hat{\nu}}\dot{\gamma}_{\hat{\mu}\hat{\alpha}}\dot{\gamma}_{\hat{\nu}\hat{\beta}} + \bar{G}(\tilde{\gamma})\gamma_{\hat{\alpha}\hat{\beta}} = 0\,, \tag{3.15}$$

where

$$\bar{G}(\tilde{\gamma}) = -4\tilde{\gamma}^{\frac{2}{3}} - \tilde{\gamma}\left(2\Lambda_4 + 3\alpha^2 + \kappa_4^2(\rho - p) + G(\tilde{\gamma})\right). \tag{3.16}$$

The trace of equations (3.15) gives

$$\left(\frac{\dot{\tilde{\gamma}}}{\tilde{\gamma}}\right)^{\cdot} + 3\bar{G}(\tilde{\gamma}) = 0\,, \tag{3.17}$$

which has a first integral

$$s \equiv \left(\frac{\dot{\tilde{\gamma}}}{\tilde{\gamma}}\right)^2 + U(\tilde{\gamma}) = constant\,, \tag{3.18}$$

where

$$U(\tilde{\gamma}) = 6\int \frac{\bar{G}(\tilde{\gamma})}{\tilde{\gamma}} d\tilde{\gamma}\,. \tag{3.19}$$

Making the conformal transformation [35]

$$\varpi_{\hat{\alpha}\hat{\beta}} \equiv \tilde{\gamma}^{-1/3}\,\gamma_{\hat{\alpha}\hat{\beta}}\,, \tag{3.20}$$

equations (3.15), due to (3.17), become

$$\ddot{\varpi}_{\hat{\alpha}\hat{\beta}} - \varpi^{\hat{\mu}\hat{\nu}}\dot{\varpi}_{\hat{\mu}\hat{\alpha}}\dot{\varpi}_{\hat{\nu}\hat{\beta}} = 0\,, \tag{3.21}$$

which are integrated to

$$\dot{\varpi}_{\hat{\alpha}\hat{\beta}} = \vartheta^{\hat{\delta}}_{\hat{\alpha}}\,\varpi_{\hat{\delta}\hat{\beta}}\,, \tag{3.22}$$

with $\vartheta^{\hat{\beta}}_{\hat{\alpha}}$ integration constants. Since $\det(\varpi_{\hat{\alpha}\hat{\beta}}) = 1$, it is $\vartheta^{\hat{\alpha}}_{\hat{\alpha}} = 0$.

We consider a diagonal metric $\gamma_{\hat{\alpha}\hat{\beta}} = diag(\gamma_{\hat{1}\hat{1}}, \gamma_{\hat{2}\hat{2}}, \gamma_{\hat{3}\hat{3}})$ and then, we have, due to the linear constraints (3.14) that $\tilde{\gamma} = \gamma^3_{\hat{3}\hat{3}} \Leftrightarrow \gamma_{\hat{1}\hat{1}}\gamma_{\hat{2}\hat{2}} = \gamma^2_{\hat{3}\hat{3}}$. System (3.22) supplies $\vartheta^{\hat{1}}_{\hat{1}}\vartheta^{\hat{2}}_{\hat{2}} \neq 0$, while all other constants $\vartheta^{\hat{\beta}}_{\hat{\alpha}}$ vanish. Thus, $\vartheta^{\hat{2}}_{\hat{2}} = -\vartheta^{\hat{1}}_{\hat{1}}$. Then, system (3.22), combined with the quadratic equation (3.8) and the first integral (3.18), supply, when (3.10) and (3.11) hold, the following algebraic relation between the constants of integration:

$$(\vartheta^{\hat{1}}_{\hat{1}})^2 = \frac{1}{3}s\,. \tag{3.23}$$

The other set of equations (3.12), (3.13) does not give compatibility. Equation (3.18), when (3.10), (3.11) hold, gives the following explicit expression for the mean expansion rate of the described universe

$$\left(\frac{\dot{\tilde{\gamma}}}{\tilde{\gamma}}\right)^2 = s + 36\tilde{\gamma}^{\frac{2}{3}} + 12\kappa_4^2\,\rho_1\,\tilde{\gamma}^{\frac{1-w}{2}} + 6\tilde{\gamma}\left(2\Lambda_4 + 3\alpha^2 \pm \right. \tag{3.24}$$
$$\left. \pm\sqrt{3}\,\alpha\sqrt{4\Lambda_4 - 2\Lambda_5 + 3\alpha^2 + 4\kappa_4^2\,\rho_1\,\tilde{\gamma}^{-\frac{1+w}{2}}}\right).$$

To conclude, after having found $\tilde{\gamma}(t)$ from (3.24), the solution of our system is

$$\gamma_{\hat{1}\hat{1}}(t) = e^{\sqrt{\frac{s}{3}}\,t}\;\tilde{\gamma}^{\frac{1}{3}}(t) \tag{3.25}$$

$$\gamma_{\hat{2}\hat{2}}(t) = e^{-\sqrt{\frac{s}{3}}\,t}\;\tilde{\gamma}^{\frac{1}{3}}(t)\,, \tag{3.26}$$

This is the unique perfect fluid solution for $n = 4$ and $E^i_j = \delta^i_j$. This solution contains 2 (positive) essential constants s, ρ_1 (plus three parameters w, Λ_4, Λ_5) and corresponds to the solution of 4-dimensional general relativity found in [36, 37] (see also [38]). The difference lies in equation (3.24) and basically in the presence of the terms multiplied by α. For $s = \Lambda_4 = \Lambda_5 = 0$, the above solution reduces to the isotropic solution found in [15]. As it is seen from (3.24), (3.25), (3.26) the solution found is always singular, while for some range of its parameters it is bounded from above. When $\tilde{\gamma}$ is extendible to infinity, equation (3.24) for $\tilde{\gamma}$ becomes in this interval the common de Sitter equation, i.e. the mean scale factor grows exponentially with proper time with an effective cosmological constant $\Lambda_4 + \frac{3}{2}\alpha^2 \pm \frac{\sqrt{3}}{2}\alpha\sqrt{4\Lambda_4 - 2\Lambda_5 + 3\alpha^2}$. Even for $\Lambda_4 = \Lambda_5 = 0$, a cosmological constant $\frac{3}{2}\alpha^2$ remains for the + sign solution, a fact first observed in [13]; for the $-$ sign, the above effective cosmological constant vanishes and the solution enters a fully 5D regime. In AdS_5 bulk with $|\Lambda_5| = \frac{2}{3}(\frac{\Lambda_4}{\alpha})^2$ ($\Lambda_4 \neq 0$) the effective cosmological constant vanishes (for suitable $\pm$ sign) and the solution enters once more a 4D regime with κ_4^2 replaced by $\kappa_4^2\,(1+\frac{3\alpha^2}{2\Lambda_4})^{-1}$. In this AdS_5 bulk we can find from equation (3.17), using also (3.11), (3.16) and (3.24), the following second order differential equation for the mean scale factor $l \equiv \tilde{\gamma}^{\frac{1}{6}}$ of the universe with respect to the proper time t_P:

$$2l^5\frac{d^2l}{dt_P^2} = -\frac{s}{9} + \alpha^2\, l^{3(1-w)}\,\left(1+\frac{2\Lambda_4}{3\alpha^2}\right)\Bigg[-(1+3w)\,\beta\,\left(1+\frac{2\Lambda_4}{3\alpha^2}\right) + l^{3(1+w)} - \\ - \frac{l^{3(1+w)}+(1-3w)\beta}{\sqrt{1+4\,\beta\, l^{-3(1+w)}}}\Bigg], \tag{3.27}$$

where $\beta \equiv \frac{\kappa_4^2\rho_1}{3\alpha^2}(1+\frac{2\Lambda_4}{3\alpha^2})^{-2} > 0$. The parameters β, Λ_4 are independent of each other since ρ_1, Λ_4 are also independent. By choosing β (for an equation of state with $w > -\frac{1}{3}$), we can make the difference of the last two terms in the above bracket positive by simply taking $l > (\frac{(1-3w)^2\beta}{2(1+3w)})^{\frac{1}{3(1+w)}}$. For such an l, say l_*, we can certainly find a Λ_4 (with $2\Lambda_4 + 3\alpha^2 > 0$) such that the whole bracket being positive; further, there exists an s (e.g. $s = 0$) such that $\frac{d^2l}{dt_P^2}|_{l_*} > 0$. Thus, we have found a brane solution which emerges as a four-dimensional general relativistic solution, possesses in the future an accelerating phase, and finally reduces to a 4D perfect-fluid solution with modified Newton's constant. Since $w > -\frac{1}{3}$, this solution does not possess event horizons. This is important in relation to the problems encountered by string theory to define a set of observable quantities analogous to an S-matrix in ordinary de Sitter spaces (see e.g. [39]).

Anisotropic solutions such as the above one, since they contain M_5 as a parameter, may serve for relating the five dimensional Planck mass to the shear/expansion parameter, using the quadrupole moment $\alpha_2 = 10^{-5}$ of the temperature pattern of the CMB radiation.

4. Conclusions

We have studied the dynamics of the most general 3-brane embedded in an $(A)dS_5$ or Minkowski bulk when the intrinsic curvature term is added in the brane action. This term is known to cause dramatic changes on the propagators studied within a fixed — not cosmological — background. We have reduced the modified dynamics of the brane to a usual Einstein dynamics coupled to a well-defined modified matter content. In particular, the total energy-momentum tensor of these equations is split into the common four-dimensional energy-momentum tensor plus additional terms containing also the bulk cosmological constant, which are all multiplied by one of the characteristic scales of the theory, i.e. $1/r_c = M_5^3/M_4^2$. These additional terms are basically square roots of the various matter terms, and can be dropped in the first era of the universe evolution characterized by some volume scale factor much smaller than a positive power of r_c. Inhomogeneous perfect fluid solutions were seen not to be necessarily incompatible with exact $(A)dS$ or Minkowski bulks, in contrast to the case of brane cosmologies not containing the ${}^{(4)}R$ term. The above form for the induced equations is useful in order to find new brane solutions following the methods of general relativity theory. As an application of this, we have found a new brane cosmological solution for an anisotropic brane, where the spacelike surfaces admit the isometry group of Bianchi type V geometry, i.e.

$$ds^2_{(4)} = -\tilde{\gamma}(t)dt^2 + \tilde{\gamma}(t)^{\frac{1}{3}}\left[e^{\sqrt{\frac{s}{3}}\,t}\,(\sigma^{\hat{1}})^2 + e^{-\sqrt{\frac{s}{3}}\,t}\,(\sigma^{\hat{2}})^2 + (\sigma^{\hat{3}})^2\right]\,, \qquad (4.1)$$

with $\tilde{\gamma}(t)$ given by equation (3.24). Choosing $s = 0$ an isotropic solution arises. Solution (4.1) approaches a known general relativistic solution in the beginning of the evolution, but finally it evolves as an inflationary anisotropic solution. There exists a fine-tuned AdS_5 bulk in which the solution eternally evolves as a usual perfect fluid solution (with non-quintessence equation of state) with no effective four-dimensional cosmological constant appearing, except that the effective Newton's constant equals the Newton's constant divided by $1 + \frac{3\alpha^2}{2\Lambda_4}$; thus, future horizons do not appear. Before this phase, there exists an accelerating era which may be in agreement with the present supernovae data [40].

Though the ${}^{(4)}R$ term arises from first order quantum corrections, it does alter the conventional behavior towards the initial singularity (whenever this encounters). Possible violations of the energy conditions may occur at later stages of the evolution. Any inhomogeneous or anisotropic solution found, based on the modified brane dynamics, would serve to investigate any possible deviations from the de Sitter future evolution, as suggested by the cosmic no-hair conjecture.

Due to the non-uniqueness of E^i_j, one could assume that the induced dynamics is not unambiguous. It is better, instead, to take the point of view that dynamics is well-defined, but the governing equations are further classified according to which class of solutions we want to pick. This classification has, of course, to do with the initial conditions of the system.

The inclusion of next order corrections in a derivative expansion of the action involves the ${}^{(5)}R^2$ term in the bulk and the ${}^{(4)}R^2$ term on the brane. One could try including these terms in an analysis of the induced brane dynamics.

In [11], except the ${}^{(4)}R$ term, two more additional terms, namely $K^\mu_\mu K^\nu_\nu$ and $K^\nu_\mu K^\mu_\nu$ proved to be necessary in the low energy effective action to reproduce the

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tree level string amplitude corresponding to scattering a massless closed string field off a bosonic Dp-brane. It would be interesting to get modified boundary equations instead of (2.11), when these two terms are present. Equations (2.11) arise alternatively, including in the action (2.1) the standard Gibbons-Hawking term [41] and making the variation with respect to g_{AB} [42, 13]. It seems, however, that the inclusion of these two additional terms does not supply a well defined variational problem.

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